MORE BIRTHDAY SURPRISES

MORTON ABRAMSON, York University, Toronto, and W. O. J. MOSER, McGill University, Montreal

There does not seem to appear in the literature a solution to the following generalization of the well-known birthday problem. Suppose a "year" has n days. If $k \geqq 1$ is given, and p people are chosen at random, what is the probability $B_k(n, p)$ that the birthdays of every pair are at least k days apart? We obtain the answer as a special case of a more general result. For other birthday problems see [1], [2]. Note that the first and last days of the year are considered to be consecutive, so that, for example, with $n=365$, December 29 and January 3 are 5 days apart. Of course

$$B_1(365, p) = (365)(364) \cdots (365 - p + 1)/(365)^p$$

is the probability that (in an ordinary year) no two of the people have the same birthday.

The birthdays of the p people are described by a p-permutation

$$b_1, b_2, \cdots, b_p \qquad b_i \in \{1, 2, \cdots, n\}, \quad b_i \neq b_j \text{ if } i \neq j. \tag{1}$$

We seek the number of permutations (1) satisfying

$$k \leqq |b_i - b_j| \leqq r, \qquad 1 \leqq i \neq j \leqq p, \tag{2}$$

and this is $p!$ times the number of p-choices

$$1 \leqq x_1 < x_2 < \cdots < x_p \leqq n \tag{3}$$

satisfying

$$k \leqq x_{i+1} - x_i, \qquad i = 1, 2, \cdots, p-1, \; x_p - x_1 \leqq r. \tag{4}$$

Corresponding to each p-choice (3) satisfying (4) there is a p-choice (obtained by setting $y_i = x_i - (i-1)(k-1)$, $i = 1, 2, \cdots, p$)

$$1 \leqq y_1 < y_2 < \cdots < y_p \leqq n - (p-1)(k-1) \tag{5}$$

satisfying

$$y_p \leqq r + y_1 - (p-1)(k-1) \tag{6}$$

(obtained from $x_p - x_1 \leqq r$ upon replacing x_p by $y_p + (p-1)(k-1)$ and x_1 by y_1). We count these in the two cases: (I) $y_1 \geqq n - r$; (II) $y_1 < n - r$. If (I) holds, (6) is redundant and the p-choices are

$$n - r \leqq y_1 < y_2 < \cdots < y_p \leqq n - (p-1)(k-1);$$

there are

$$\binom{n - (p-1)(k-1) - n + r + 1}{p} \tag{7}$$

of these. If (II) holds, the p-choices satisfy

$$1 \leqq y_1 < n - r, \qquad y_1 < y_2 < \cdots < y_p \leqq r + y_1 - (p-1)(k-1);$$

there are

$$(n - r - 1)\binom{r + y_1 - (p-1)(k-1) - y_1}{p-1} \tag{8}$$

of these. Adding (7) and (8) we have the

$$\frac{n - (p-1)(k-n+r)}{p}\binom{r - (k-1)(p-1)}{p-1} \tag{9}$$

p-choices (3) satisfying (4). (This proof has been included to make this note self-contained; the result is contained in [**3**; formula (37)′] and [**4**; formula (19) with l' large and $l=n-r$].) Hence,

$$\frac{p!}{n^p}\cdot\frac{n - (p-1)(k-n+r)}{p}\binom{r - (k-1)(p-1)}{p-1} \tag{10}$$

is the probability that the birthdays (1) satisfy (2). With $r=n-k$ in (10) we have

$$B_k(n, p) = \frac{(p-1)!}{n^{p-1}}\binom{n - p(k-1) - 1}{p-1}.$$

The values of $1-B_k(365, p)$ are listed for some values of p and k. For example, $1-B_2(365, p)$ is the probability that (in an ordinary year) some pair have the same or adjacent birthdays. Letting $s(k)$ denote the smallest p for which $1-B_k(365, p)\geqq\frac{1}{2}$ we see the surprisingly small number of people for which, with probability $\geqq\frac{1}{2}$, some pair have birthdays less than k days apart.

$1 - B_k\ (365, p)$

p \ k	1	2	3	4
5	.02713	.07971	.13013	.17844
8	.07433	.20873	.32604	.42812
9	.09462	.26042	.39901	.51433
10	.11694	.31472	.47209	.59648
11	.14114	.37056	.54328	.67210
12	.16702	.42693	.61090	.73952
13	.19440	.48287	.67363	.79778
14	.22310	.53749	.73053	.84663
21	.44368	.83603	.95537	.98890
22	.47569	.86378	.96774	.99313
23	.50729	.88791	.97709	.99586
24	.53834	.90864	.98401	.99757

k	$s(k)$
1	23
2	14
3	11
4	9
5	8
6	8
7	7
8	7
9	6
10	6

References

1. Robert E. Greenwood, and Arthur Richert, Jr., A birthday holiday problem, J. Combinatorial Theory, 5 (1968) 422–424.
2. M. S. Klamkin and D. J. Newman, Extensions of the birthday surprise, J. Combinatorial Theory, 3 (1967) 279–282.
3. R. Lagrange, Sur les combinaisons d'objets numérotés, Bull. Sci. Math., 87 (1963) 29–42.
4. W. O. J. Moser and Morton Abramson, Enumeration of combinations with restricted differences and cospan, J. Combinatorial Theory, 7 (1969) 162–170.

ON A PROBLEM OF J. C. MOORE

ALEX HELLER, City University of New York

J. C. Moore has proposed an interesting problem concerning the "contractibility" of continuous maps.

If X and Y are topological spaces, a map $f:X\to Y$ is *contractible* if X and Y have contracting homotopies $\xi_t:X\to X$ and $\eta_t:Y\to Y$ (i.e., homotopies of the identity with a constant map) such that $f\xi_t=\eta_t f$ for all t. Thus a contractible map has contractible domain and range. Moore asks whether the converse is true.

The following picture may suggest what is involved:

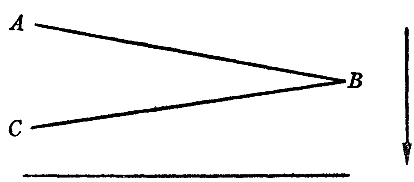

FIG. 1

The map is the projection indicated by the arrow. It is contractible: AB and CB need only be shrunk simultaneously to B.

Moore further suggests that the map $f:X\to Y$ indicated by the following picture:

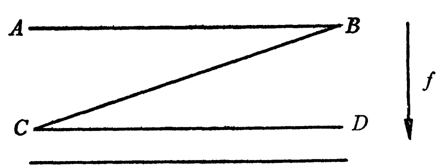

FIG. 2

is *not* contractible.

This suggestion seems plausible: it is hard to see how the homotopy ξ_t can deform the image of X away from A and D while B and C are covered. But