MORE BIRTHDAY SURPRISES

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There does not seem to appear in the literature a solution to the following generalization of the well-known birthday problem. Suppose a “year” has \(n\) days. If \(k \geq 1\) is given, and \(p\) people are chosen at random, what is the probability \(B_k(n, p)\) that the birthdays of every pair are at least \(k\) days apart? We obtain the answer as a special case of a more general result. For other birthday problems see [1], [2]. Note that the first and last days of the year are considered to be consecutive, so that, for example, with \(n = 365\), December 29 and January 3 are 5 days apart. Of course

\[
B_1(365, p) = \frac{(365)(364) \cdots (365 - p + 1)}{(365)^p}
\]

is the probability that (in an ordinary year) no two of the people have the same birthday.

The birthdays of the \(p\) people are described by a \(p\)-permutation

\[
b_1, b_2, \ldots, b_p \quad b_i \in \{1, 2, \ldots, n\}, \quad b_i \neq b_j \text{ if } i \neq j.
\]

We seek the number of permutations (1) satisfying

\[
k \leq |b_i - b_j| \leq r, \quad 1 \leq i \neq j \leq p,
\]

and this is \(p!\) times the number of \(p\)-choices

\[
1 \leq x_1 < x_2 < \cdots < x_p \leq n
\]

satisfying

\[
k \leq x_{i+1} - x_i, \quad i = 1, 2, \ldots, p - 1, \quad x_p - x_1 \leq r.
\]

Corresponding to each \(p\)-choice (3) satisfying (4) there is a \(p\)-choice (obtained by setting \(y_i = x_i - (i - 1)(k - 1), i = 1, 2, \ldots, p\))

\[
1 \leq y_1 < y_2 < \cdots < y_p \leq n - (p - 1)(k - 1)
\]

satisfying

\[
y_p \leq r + y_1 - (p - 1)(k - 1)
\]

(obtained from \(x_p - x_1 \leq r\) upon replacing \(x_p\) by \(y_p + (p - 1)(k - 1)\) and \(x_1\) by \(y_1\)). We count these in the two cases: (I) \(y_1 \geq n - r\); (II) \(y_1 < n - r\). If (I) holds, (6) is redundant and the \(p\)-choices are

\[
n - r \leq y_1 < y_2 < \cdots < y_p \leq n - (p - 1)(k - 1);
\]

there are

\[
\binom{n - (p - 1)(k - 1) - n + r + 1}{p}
\]

of these. If (II) holds, the \(p\)-choices satisfy
1 \leq y_1 < n - r, \quad y_1 < y_2 < \cdots < y_p \leq r + y_1 - (p - 1)(k - 1);

there are

\begin{equation}
(n - r - 1) \binom{r + y_1 - (p - 1)(k - 1) - y_1}{p - 1}
\end{equation}

of these. Adding (7) and (8) we have the

\begin{equation}
\frac{n - (p - 1)(k - n + r)}{p} \binom{r - (k - 1)(p - 1)}{p - 1}
\end{equation}

\(p\)-choices (3) satisfying (4). (This proof has been included to make this note self-contained; the result is contained in [3; formula (37')] and [4; formula (19) with \(l'\) large and \(l = n - r\).) Hence,

\begin{equation}
\frac{p!}{n^p} \frac{n - (p - 1)(k - n + r)}{p} \binom{r - (k - 1)(p - 1)}{p - 1}
\end{equation}

is the probability that the birthdays (1) satisfy (2). With \(r = n - k\) in (10) we have

\[ B_k(n, p) = \frac{(p - 1)!}{n^{p-1}} \binom{n - p(k - 1) - 1}{p - 1}. \]

The values of \(1 - B_k(365, p)\) are listed for some values of \(p\) and \(k\). For example, \(1 - B_2(365, p)\) is the probability that (in an ordinary year) some pair have the same or adjacent birthdays. Letting \(s(k)\) denote the smallest \(p\) for which \(1 - B_k(365, p) \geq \frac{1}{2}\) we see the surprisingly small number of people for which, with probability \(\geq \frac{1}{2}\), some pair have birthdays less than \(k\) days apart.

\[
\begin{array}{|c|cccc|c|c|}
\hline
p & 1 & 2 & 3 & 4 & k & s(k) \\
\hline
5 & .02713 & .07971 & .13013 & .17844 & 1 & 23 \\
8 & .07433 & .20873 & .32604 & .42812 & 2 & 14 \\
9 & .09462 & .26042 & .39901 & .51433 & 3 & 11 \\
10 & .11694 & .31472 & .47209 & .59648 & 4 & 9 \\
11 & .14114 & .37056 & .54328 & .67210 & 5 & 8 \\
12 & .16702 & .42693 & .61090 & .73952 & 6 & 8 \\
13 & .19440 & .48287 & .67363 & .79778 & 7 & 7 \\
14 & .22310 & .53749 & .73053 & .84663 & 8 & 7 \\
21 & .44368 & .83603 & .95537 & .98890 & 9 & 6 \\
22 & .47569 & .86378 & .96774 & .99313 & 10 & 6 \\
23 & .50729 & .88791 & .97709 & .99586 & 10 & 6 \\
24 & .53834 & .90864 & .98401 & .99757 & 10 & 6 \\
\hline
\end{array}
\]
References


ON A PROBLEM OF J. C. MOORE

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J. C. Moore has proposed an interesting problem concerning the "contractibility" of continuous maps.

If $X$ and $Y$ are topological spaces, a map $f:X \to Y$ is contractible if $X$ and $Y$ have contracting homotopies $\xi_t:X \to X$ and $\eta_t:Y \to Y$ (i.e., homotopies of the identity with a constant map) such that $f\xi_t=\eta_tf$ for all $t$. Thus a contractible map has contractible domain and range. Moore asks whether the converse is true.

The following picture may suggest what is involved:

![Fig. 1](image1)

The map is the projection indicated by the arrow. It is contractible: $AB$ and $CB$ need only be shrunk simultaneously to $B$.

Moore further suggests that the map $f:X \to Y$ indicated by the following picture:

![Fig. 2](image2)

is not contractible.

This suggestion seems plausible: it is hard to see how the homotopy $\xi_t$ can deform the image of $X$ away from $A$ and $D$ while $B$ and $C$ are covered. But